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HALF-SPACE THEOREMS FOR MINIMAL SURFACES WITH BOUNDED CURVATURE

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Abstract

First we prove a version of the Strong Half-Space Theorem for minimal surfaces with bounded curvature in $\mathbb{R}^3$. With the techniques developed in our proof we give criteria for deciding if a complete minimal surface is proper. We prove a mixed version of the Strong Half-Space Theorem. Turning to 3-dimensional manifolds of bounded geometry and positive Ricci curvature, we show that complete injectively immersed minimal surfaces with bounded curvature are proper and as a corollary we have a Half-Space Theorem in this setting. Finally we show an application of the maximum principle for nonproper minimal immersions in $\mathbb{R}^3$.

1. Introduction

The Strong Half-Space Theorem [8], states that two complete, minimally and properly immersed surfaces in $\mathbb{R}^3$ intersect unless they are parallel planes. The word strong there stands in opposition to weak in the version where one of the surfaces is a plane and will be refered as Half-Space Theorem.

There is an extension, due to Anderson and Rodriguez [2], of the Strong Half-Space Theorem that shows that in a complete oriented noncompact 3-dimensional Riemannian manifold N with nonnegative Ricci curvature $\mathrm{Ric}_N \geq 0$ and sectional curvature bounded from above $K_N \leq b$, any two complete properly immersed oriented minimal surfaces, intersect unless they are totally geodesic and parallel leaves in a local product structure. In all these results, properness is required.

On the other hand, Xavier [16] proved a version of the (weak) Half-Space Theorem where instead of properly immersed, he required

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bounded curvature. He showed that the convex hull of a complete nonplanar minimal surface with bounded curvature in $\mathbb{R}^3$ is all of $\mathbb{R}^3$. In this paper, we prove an extension of Xavier's Half-Space Theorem in the same way that the Strong Half-Space extends the weak Half-Space Theorem. We prove the following theorem:

Theorem 1.1. *Let M_1 and M_2 be complete minimal immersed surfaces in $\mathbb{R}^3$ with bounded curvature. Then $M_1 \cap M_2 \neq \emptyset$ unless they are parallel planes.*

This result raises the problem of whether the hypothesis of properly immersed in minimal surface theory can be replaced by geometric hypotheses. One may take two points of view in looking at this problem. First, one can try to prove theorems that substitute geometric hypotheses for the hypothesis of proper immersion. Theorem 1.1 fits this point of view. Second, one can look for geometric hypotheses that imply that a complete minimal surface is proper. We should remark that, requiring bounded curvature for a complete, minimally immersed surface, does not guarantee its properness. For instance, in [3] Andrade constructs a complete immersion of the plane $\mathbb{C}$ into $\mathbb{R}^3$ with bounded curvature, dense in a proper and unbounded subset with nonempty interior of $\mathbb{R}^3$. See also [12] for another example.

The ideas developed in the proof of Theorem 1.1 can be applied to prove theorems in the spirit of the second point of view, about complete minimal surfaces with boundary and bounded curvature. Here, for a complete surface with boundary, we understand a surface with boundary where all Cauchy sequences converge. We prove the following theorem:

Theorem 1.2. *Let $\varphi : M \hookrightarrow \mathbb{R}^3$ be a complete minimally immersed surface with boundary ∂M (possibly empty) and bounded curvature such that $\varphi(M) \subset \overline{\Omega}$, where Ω is a mean convex domain. If $\partial M \neq \emptyset$, we suppose that $\varphi\,|_{\partial M}\colon \partial M \hookrightarrow \partial\Omega \subset \mathbb{R}^3$ is proper. Then one of the following conditions holds:*

- i) *φ is proper.*
- ii) *The limit set* Lim φ, *(see Definition 2.2), is a union of parallel planes lying in the interior of a slab or in a half-space inside $\overline{\Omega}$.*

In both cases, there are planes separating $\partial\,\Omega$ from Lim φ*, unless $\partial\Omega$ is a plane contained in the limit set.*

As a corollary of Theorem (1.2) we have the following criteria for deciding whether a complete minimal surface of bounded curvature is proper.

Corollary 1.3. *Let* Σ *be a complete nonflat minimal surface in* $\mathbb{R}^3$*. Let* $\varphi : M \hookrightarrow \mathbb{R}^3$ *be a complete minimal immersion with bounded curvature transversal to* Σ*. Set* $\Gamma = \varphi^{-1}(\Sigma)$*. Assume that* $\varphi \mid_{\Gamma} : \Gamma \hookrightarrow \mathbb{R}^3$ *is proper. Suppose that one of the following conditions holds:*

i) Σ *is proper.*

ii) Σ *has bounded curvature.*

Then φ *is proper.*

In particular, we have the following Mixed Half-Space Theorem.

Corollary 1.4 (Mixed Half-Space Theorem). *Let* M_1 *be a complete proper minimal surface and* M_2 *be a complete minimal surface with bounded curvature in* $\mathbb{R}^3$*. Then* $M_1 \cap M_2 \neq \emptyset$*, unless they are parallel planes.*

We now turn to complete 3-dimensional Riemannian manifolds with bounded geometry and positive Ricci curvature. Bounded geometry here means sectional curvature bounded from above and injectivity radius bounded away from zero.

Theorem 1.5. *Let* $\varphi : M^n \hookrightarrow N^{n+1}$ *be a complete minimal immersed hypersurface with scalar curvature bounded from below in a complete dimensional Riemannian manifold* N *of bounded geometry. Suppose in addition that* N *has nonnegative Ricci curvature* $\mathrm{Ric}_N \geq 0$*. Then* φ *is proper or every orientable leaf* $S \subset \mathrm{Lim}\,\varphi$ *such that* $S \cap \varphi(M) = \emptyset$ *is stable. Moreover, if* S *is compact then* S *is totally geodesic and the Ricci curvature of* N *is identically zero in the normal directions to* S*.*

Corollary 1.6. *Let* $M \subset N$ *be a complete oriented injectively and minimally immersed surface with bounded curvature in a* 3*-dimensional Riemannian manifold of bounded geometry and positive Ricci curvature. Then:*

1. *If* N *is compact then* M *is compact.*
2. *If* N *is not compact then* M *is proper.*

Corollary (1.6) together with the Anderson-Rodriguez Half-Space Theorem yield the following theorem.

Theorem 1.7. *Let* M_1 *and* M_2 *be complete, minimally and injectively immersed surfaces with bounded sectional curvature in a complete,* *noncompact* 3*-dimensional Riemannian manifold of bounded geometry and positive Ricci curvature. Then* $M_1 \cap M_2 \neq \emptyset$*, unless they are both totally geodesic and parallel leaves in a local product structure.*

Finally we present an alternate proof of a result that is a direct corollary of Theorem (1.1) because it shows an application of the maximum principle for nonproper minimal immersions.

Let M *be a complete minimal surface of* $\mathbb{R}^3$ *with bounded curvature and let* C *be a catenoid. Then* $M \cap C \neq \emptyset$*.*

H. Rosenberg independently has proven Theorem 1.1 and Corollary 1.6, (1) (see [13]). In proving a Half-Space Theorem in $\mathbb{R}^3$, one follows the same idea as to prove Hoffman-Meeks Strong Half-Space Theorem, i.e., one needs to construct a complete and stable minimal surface separating M_1 and M_2. For this, one constructs mean convex barriers, and our proof differs from Rosenberg's in the way these barriers are constructed. Theorem 1.1 and Corollary 1.6 were divulged by the second author in the regular seminar at IMPA-Rio de Janeiro in November 1998.

2. Half-Space Theorem (Theorem 1.1)

This section is divided in two subsections. In the first subsection, we construct a mean convex barrier and in the second one we present the proof of Theorem 1.1.

2.1 Barrier construction

Throughout this section we denote by B_R an open Euclidean ball with radius R centered at the origin and when it has another center p we write $B(p,R)$ instead. For a set A we will denote by $\overline{A}$ its closure in $\mathbb{R}^3$ and by $T_\epsilon(A) = \{p \in \mathbb{R}^3\,;\, \mathrm{dist}(p,A) \leq \epsilon\}$ its closed ϵ-tubular neighborhood. Let us consider $M \subset \mathbb{R}^3$ a complete nonproper minimal surface with bounded Gaussian curvature such that $\overline{M} \neq \mathbb{R}^3$. Set $r_0 < (\sup_{x\in M} \sqrt{|k_M(x)|})^{-1}$, where $k_M(x)$ is the Gaussian curvature of M at x in such a way that $T_{r_0}(\overline{M})$ is not all of $\mathbb{R}^3$. To construct the barrier we will need some results from [7] about *sets with positive reach* and the *catograph set of a function* . The reach of a subset $A \subset \mathbb{R}^3$ is the largest ϵ (possibly ∞) such that if $x \in \mathbb{R}^3$ and the distance $d(x,A)$

is smaller than ϵ, then A contains a unique point $\bar{x}$ nearest to x. If $f : A \to \mathbb{R}$ is a function, then the *catograph set* of f is defined as $U = \{(x, y) \in A \times \mathbb{R}\,; 0 \leq y \leq f(x)\}$.

Lemma 2.1. *The tubular neighborhood $T_r(\overline{M})$, for a.a. $0 < r \leq r_0$, has Lipschitz boundary and its complement $\overline{\mathbb{R}^3 \setminus T_r(\overline{M})}$ has positive reach. Also every boundary point of $T_r(\overline{M})$ has inner support spheres of radius t, for $0 < t \leq r$. In particular, the outside tangent cone of $\partial T_r(\overline{M})$ has no angle bigger than π.*

Proof. That the tubular neighborhood $T_r(X)$ has Lipschitz boundary and its complement $\overline{\mathbb{R}^3 \setminus T_r(X)}$ has positive reach are proved in [7] (see Main Theorem) by J. Howland Fu for X bounded. Taking $X = M \cap \overline{B_R}$ and making R going to infinity we conclude that $\partial T_r(M)$ is Lipschitz.

For each $p \in \partial T_r(M)$ there exists at least a point $p_0 \in \overline{M}$ such that $\operatorname{dist}(p, \overline{M}) = |p - p_0|$. Observe that $p \in \partial B(p_0, r)$ and $B(p_0, r) \subset T_r(M)$. Thus $\partial B(p_0, r)$ is a support at p for $\partial T_r(M)$ inside $T_r(M)$. For any point p' in the interior of the line segment $[p_0, p]$, the ball $B(p', |p' - p|)$ touches $\partial T_r(M)$ at exactly one point. Moving p on $\partial T_r(M)$ keeping $|p' - p|$ constant, we get that the reach of $\overline{\mathbb{R}^3 \setminus T_r(M)}$ is no less than $|p' - p|$ or even better, than r.

Now assuming that the tangent cone of $\partial T_r(M)$ is not a plane, we may have more than one point $p_0 \in \overline{M}$ realizing the distance to p. But for each of such points we get a support sphere for $\partial T_r(M)$ at p inside $T_r(M)$. In particular, the tangent cone is inside the cone determined by the intersection of these spheres at p. Therefore no plane section has angle bigger than π.

We will need a basic lemma (Lemma 2.3) about limit sets of isometric immersions $\varphi : M^m \hookrightarrow N^n$, developed in [4] in more general situation than the one considered here. For completeness, we first give the definition of limit sets.

Definition 2.2. Let $\varphi : M^m \hookrightarrow N^n$, $1 \leq m < n$, be an isometric immersion where M and N are complete Riemannian manifolds of dimensions m and n, respectively. The limit set of φ, denoted $\operatorname{Lim} \varphi$, is the following set:

$$\begin{aligned}\operatorname{Lim}\varphi &= \{p \in N\,;\ \exists\,\{p_n\} \subset M,\ \operatorname{dist}_M(p_0,p_n) \to \infty \\ &\qquad\qquad\qquad\qquad \text{and } \operatorname{dist}_N(p,\varphi(p_n)) \to 0\} \\ &= \bigcap_{K \subset M} \overline{\varphi(M) \setminus \varphi(K)},\ K \text{ compact}.\end{aligned}$$

Observe that $\operatorname{Lim}\varphi \subset \overline{\varphi(M)}$ is a closed set and $\operatorname{Lim}\varphi = \emptyset$ if and only if φ is proper. Sometimes when the immersion is not explicitly presented (i.e., $M \subset N$) we denote the limit set by $\operatorname{Lim} M$. We have the following lemma.

Lemma 2.3. *Let $M \subset \mathbb{R}^3$ be a complete nonproper minimal surface with bounded curvature and let $p \in \operatorname{Lim} M$. Then there exists a sequence of minimal disks $D_i \subset M$ converging uniformly (in the C^∞-topology) to a minimal disk $D \subset \operatorname{Lim} M$ containing p. Moreover, the limit disk D can be extended to a complete minimal surface $S_p \subset \operatorname{Lim} M$ passing through p with bounded curvature. If the limit disk D is flat, then S_p is a plane.*

This lemma is a consequence of the fact that each point $x \in M$ has a neighborhood V_x that can be graphed over a ball in the tangent plane of M at x with radius uniformly bounded from below, coupled with convergence results of minimal graphs.

Lemma 2.4. *Let $M \subset \mathbb{R}^3$ be a complete nonproper minimal surface with bounded curvature. Let us assume that no limit disk given by Lemma (2.3) is flat. Given $p \in \partial T_r(\overline{M})$, $0 < r < r_0$, then there is a catograph set U_p of some function f such that:*

(i) *$p \in \operatorname{Int}(U_p)$.*

(ii) *$S_p = (\partial U_p) \setminus T_r(\overline{M})$ is a compact embedded surface with nonnegative mean curvature for unit normal vector pointing outside U_p and $\partial S_p \subset \partial T_r(\overline{M})$.*

(iii) *For $p, q \in \partial T_r(\overline{M})$ with the corresponding surfaces S_p and S_q intersecting in p' interior to both surfaces, the outside angle of $U_p \cap U_q$ at p' is at most π.*

Proof. Let $p \in \partial T_r(\overline{M})$ and $p_0 \in \overline{M}$ as before. There exists a sequence of minimal embedded disks $D_k \subset M$ converging uniformly to a minimal embedded disk $D_0 \subset \overline{M}$ containing p_0. It is clear that

$D_0 \cap B(p,r) = \emptyset$, otherwise there would exist a point $q \in D_0$ such that $\operatorname{dist}(p,q) < r$ contradicting the fact that $p \in \partial T_r(\overline{M})$. Now let ν be a continuous unit normal vector field on D_0 so that $\nu(p_0)$ is pointing toward p. We may assume that the image of D_0 by ν takes the value $\nu(p_0)$ (in the unit sphere) only once with multiplicity. Otherwise, D_0 would be a flat disk. The parallel disks $D_t = \{x(t) := x + t\nu(x); x \in D_0\}$, $t \leq r$, are well defined and embedded provided that $r \leq r_0$. Its mean curvature H_t is given by

$$H_t(x(t)) = \frac{-k_M(x)\, t}{1 + k_M(x)\, t^2} \geq 0, \quad t \geq 0. \tag{1}$$

The line segment $[p_0, p] = \{p_0 + t\nu(p_0),\ 0 \leq t \leq r\}$ is perpendicular to $\partial\, \overline{B(p_0, r)}$ at p and to the tangent space $T_{p_0} D_0$. Let C be the solid cylinder with axis generated by $\nu(p_0)$ and orthogonal cross section the disk $B = B(p_0, \epsilon) \cap T_{p_0} D_0$. We may assume that ∂D_t, $(0 < t \leq r)$ is outside C and $D_t \cap C$ is a graph over B. The surface D_r is also a support for $\partial T_r(\overline{M})$ at p.

Fix t_0, $0 < t_0 < r$ and observe that $D_{t_0} + (r - t_0)\nu(p_0)$ is contained in $T_r(\overline{M})$ with the boundary in the interior. To see this take $q \in D_{t_0} \cap C$ and $q' \in D_0$ the nearest point to q. Then $|q' - (q + (r - t_0)\nu(p_0))| < |q' - q| + r - t_0 = r$. The inequality is strict because $(|q' - q| = t_0)$ and $\nu(q') \neq \nu(p_0)$. Hence, $D_{t_0} + (r - t_0)\nu(p_0)$ lies in one side of D_r and touches D_r at p. We can choose $\delta > 0$ so that:

(i) The disk $D_{t_0} \cap C + (\delta + r - t_0)\nu(p_0)$ crosses the boundary $\partial T_r(\overline{M})$, dividing it in at leas t two sets, one of them being a small disk with p in the interior.

(ii) The boundary of $D_{t_0} \cap C + (\delta + r - t_0)\nu(p_0)$ is contained in the interior of $T_r(\overline{M})$.

Writing $D_{t_0} \cap C + (\delta + r - t_0)\nu(p_0)$ as graph of a function f over B, we set U_p as the *catograph set* of f, that is, $U_p = \{(x, y) \in C; 0 \leq y \leq f(x)\}$. The assertion (iii) follows from this construction.

Conclusion 2.5. Let $\mathbb{R}^3 \setminus \overline{M} \neq \emptyset$ and $\overline{M} \cap B_R \neq \emptyset$. Let $X \subset \overline{B_R}$ be a closed set not intersecting $\overline{M}$. Then there is an $\epsilon = \epsilon(R, M)$ and a C^∞ by parts surface S_ϵ such that:

(i) $S_\epsilon \subset T_{2\epsilon}(\overline{M}) \setminus T_\epsilon(\overline{M})$.

(ii) $S_\epsilon \cap T_{2\epsilon}(X) = \emptyset$.

(iii) S_ϵ is part of the boundary isolating $\overline{M}$ and $T_{2\epsilon}(X)$ and is mean convex with respect to the open set between them.

Remark 2.6. These surfaces S_ϵ can be made by minimal disks. Just consider $t_0 = 0$ in their construction as above. One has to show that in that case we still have $D_0 + r\nu(p_0)$ contained in $T_r(\overline{M})$ with the boundary in the interior. But this is true, for if we take a point $p_0 \neq q \in D_0 \cap C$, the distance between $q + r\nu(p_0)$ and D_0 is r if and only if $\nu(q) = \nu(p_0)$. The rest of the construction is the same.

Remark 2.7. Conclusion 2.5 is true if we replace $\overline{M}$ by $\operatorname{Lim} M$.

2.2 Proof of Theorem 1.1

Let $M_i \subset \mathbb{R}^3$, $i = 1, 2$ be two complete nonproper minimal surfaces with bounded Gaussian curvature.

1^{st}. Assume that $\overline{M_1} \cap \overline{M_2} \neq \emptyset$ and $M_1 \cap M_2 = \emptyset$. Take a point $p \in \overline{M_1} \cap \overline{M_2}$. There are two sequences of minimal disks $D_k^i \subset M_i$ $i = 1, 2$ converging uniformly to minimal disks $D^i \subset \overline{M_i}$, $i = 1, 2$, both containing p. D^1 and D^2 can be extended to complete minimal surfaces with bounded curvature $S_p^1 \subset \overline{M_1}$ and $S_p^2 \subset \overline{M_2}$ respectively. They can not intersect themselves transversally, otherwise M_1 would intersect M_2. Therefore they are tangent to each other at p and lie to one side of the common tangent plane at p. Then by the maximum principle $S_p^1 = S_p^2 := S_p$. Clearly S_p does not intersect M_i, $i = 1, 2$, nor has self-intersections because we would have $M_1 \cap M_2 \neq \emptyset$. By Theorem 1.5 below, S is stable and by [5] or [6], S is a plane. S separates M_1 from M_2 or they are at the same side of S. In both cases, by Xavier's Half-Space Theorem, M_1 and M_2 are parallel planes. This contradicts the nonproperness assumption.

2^{nd}. Suppose that $\overline{M_1} \cap \overline{M_2} = \emptyset$ and $\overline{M_1}$, contains a flat limit disk D. Then D can be extended to a plane S. By Xavier' s Half-Space Theorem M_2 intesects S and thus M_1, unless M_2 is a plane parallel to S. This contradicts the hypotheses that M_2 is nonproper and M_2 does not intersect M_1.

3^{rd}. Assume now that $\overline{M_1} \cap \overline{M_2} = \emptyset$ and $\overline{M_i}$, $i = 1, 2$ has no flat limit disk. Now let $R_0 > 0$ be such that B_{R_0} intersects $\overline{M_i}$. Taking X as $\overline{M_i} \cap \overline{B_{R_0}}$, for $i = 1, 2$, we find $\epsilon_0 = \epsilon_0(M_i, R_0)$ such that the surfaces

$S_{\epsilon\, i}$ of Conclusion 2.5 together with parts of $\partial\, B_{R_0}$ are the boundary of a mean convex set $\Omega(R_0)$ isolating $T_{\epsilon_0}(\overline{M_1})$ and $T_{\epsilon_0}(\overline{M_2})$ inside $\overline{B_{R_0}}$. Let $x_1 \in \overline{M_1} \cap \overline{B_{R_0}}$ and $x_2 \in \overline{M_2} \cap \overline{B_{R_0}}$ realize the distance of these two sets. The line segment $[x_1, x_2]$ intersects $\overline{M_1}$ and $\overline{M_2}$ only at the end points. Consider $\Omega(R_0)$ with the usual orientation and $[x_1, x_2]$ oriented from x_1 to x_2. Let S be the connected component of $S_{\epsilon\, 1}$ crossing $[x_1, x_2]$. By construction the intersection number of this set is exactly one.

Suppose there is a closed curve $\gamma \subset S$ homotopic to a point in $\Omega(R_0)$. By [10] (or [15]), one has that γ bounds an embedded minimal disk D_γ inside $\Omega(R_0)$. Cut S and $\Omega(R_0)$ along D_γ and glue two copies of D_γ along the boundary. This surgery does not change the intersection number with $[x_1, x_2]$. It also produces a new mean convex domain. Doing this a finite number of times we get an incompressible surface $S(R_0)$ in a new mean convex domain $\widetilde{\Omega}(R_0)$ crossing $[x_1, x_2]$. We claim that $S(R_0)$ has boundary and, of course $\partial\, S(R_0) \subset \partial\, B_{R_0}$. If not, $S(R_0)$ bounds $\overline{M_1}$ or $\overline{M_2}$ once they are on opposite sides. But there are no complete bounded minimal surfaces in $\mathbb{R}^3$ with bounded Gaussian curvature (see [9]). Hence using again [10] or [15] we get an stable minimal surface $S_0 \subset \overline{B_{R_0}}$ such that:

(i) $\partial\, S_0 = \partial\, S(R_0)$ in $\partial\, B_{R_0}$.

(ii) S_0 is homotopic to $S(R_0)$ in $\widetilde{\Omega}(R_0)$.

(iii) The intersection number of S_0 and $[x_1, x_2]$ is one.

(iv) S_0 separates points of $\overline{M_1}$ and $\overline{M_2}$ inside B_{R_0}.

Take a divergent increasing sequence of R_j and corresponding $S_j \subset B_{R_j}$ stable minimal surfaces satisfying (i)–(iv). It is well known that S_j converges in compact parts to a complete stable minimal surface S immersed in $\mathbb{R}^3$ (see [1]), and separating points of $\overline{M_1}$ and $\overline{M_2}$, (x_1 and x_2, for instance). By [5] or [6], S is a plane and $\overline{M_1}$ and $\overline{M_2}$ are on opposite sides of this plane S. By Xavier's Half-Space Theorem [16], M_1 and M_2 are parallel planes, contradicting the nonproperness assumption.

To finish the proof of Theorem (1.1) we need to consider the case that one of the surfaces is proper with bounded curvature. This is done in the next section in a more general context, i.e., one of the surfaces is proper regardless the bounds on the curvature.

3. Properness criteria

Here we present the proof of Theorem 1.2 and its corollaries. Let Ω be a mean convex domain with boundary $\Sigma = \partial\,\Omega$. By definition of mean convex sets, Σ is a union of pieces of regular surfaces with nonnegative mean curvature with respect to normal vector field pointing toward the interior of Ω, glued by their boundaries with inner angle less than or equal to π. If a proper minimal surface M inside Ω touches one face Σ' at an interior point x_0, then by the maximum principle Σ' is contained in M. If $x_0 \in \partial\,\Sigma'$, then x_0 is also in the boundary of a neighbor face Σ'' and these faces are tangent to M. In a similar way, (by the maximum principle at the boundary), we can conclude that Σ' and Σ'' are contained in M, if M is large enough. Moreover, if Σ has a compact component, then Ω is compact. Further, if there is a plane $\mathbb{P}_0 \subset \Omega$ and if we let $\mathbb{P}_0' \subset \Omega$ be a plane parallel to $\mathbb{P}_0$ closest to Σ, then $\mathbb{P}_0' \cap \Sigma \neq \emptyset$, implies that $\mathbb{P}_0' \subset \Sigma$.

3.1 Proof of Theorem 1.2

Now suppose that Ω is a mean convex domain with boundary $\Sigma = \partial\,\Omega$ and $\varphi : M \hookrightarrow \mathbb{R}^3$ is a complete nonproper minimally immersed surface with boundary and bounded curvature such that $\varphi(M) \subset \overline{\Omega}$. Assume that if $\partial M \neq \emptyset$, then $\varphi\,|_{\partial M}\colon \partial M \hookrightarrow \partial\,\Omega$ is proper. Hence, there is no divergent sequence $\{x_k\}$ in M with $\{\varphi(x_k)\}$ accumulating in Σ. It follows that the limit set Lim φ exists and it is a union of complete immersed minimal surfaces with bounded curvature inside Ω.

Lemma 3.1. *Under the above condition, there exists a plane* $\mathbb{P}_0$ *separating* Lim φ *from* $\partial\,\Omega$*, unless* $\partial\Omega$ *is a plane* $\mathbb{P}_0 \subset$ Lim φ*. Moreover all* S_p *in* Lim φ *are planes.*

Proof. If Lim $\varphi \cap \partial\Omega \neq \emptyset$ then $\partial\Omega$ is a leaf from Lim φ. By a small modification of Theorem 1.5 we have that $\partial\Omega$ is stable and therefore a plane. Otherwise, take a ball B_R intersecting Lim φ and $\partial\Omega$. Choose a point $x_1 \in$ Lim φ and $x_2 \in \partial\Omega$ nearest to x_1. Without loss of generality, we may assume that the line segment $L = [x_1, x_2]$ from x_1 to x_2 lies inside Ω and does not intersect Lim φ and $\partial\Omega$ at interior points of L. Following the proof of Theorem 1.1, we can construct a plane $\mathbb{P}_0$ in Ω intersecting L and separating Lim φ from $\partial\,\Omega$. Since S_p, for $p \in$ Lim φ, has bounded curvature, by Xavier's Half-Space Theorem S_p must be a plane parallel to $\mathbb{P}_0$. Therefore, for each point $p \in$ Lim φ, $S_p \subset$ Lim φ is a plane parallel to $\mathbb{P}_0$.

Lemma 3.1 finishes the proof of Theorem 1.2.

3.2 Corollaries 1.3 & 1.4

Suppose that $\varphi : M \hookrightarrow \mathbb{R}^3$ is complete nonproper minimal immersion with bounded Gaussian curvature and $\Gamma = \varphi^{-1}(\Sigma)$ is proper.

Case i). $\mathbb{R}^3 \setminus \Sigma$ is a union of mean convex domains. Choose a complete noncompact connected component M' of $M \setminus \Gamma$. Suppose that $\varphi\mid_{M'}: M' \hookrightarrow \mathbb{R}^3$ is not proper. Then by Theorem 1.2, there is a plane separating the limit set of $\varphi\mid_{M'}$ from Σ. By Xavier's Half-Space Theorem Σ is a plane, a contradiction to the hypothesis. Thus the restriction of φ to any noncompact component M' of $M \setminus \Gamma$ is proper and $\varphi : M \hookrightarrow \mathbb{R}^3$ itself is proper.

Case ii). Suppose that φ is not proper. Observe that $\varphi(M) \cap \overline{\Sigma} = \varphi(\Gamma)$. For, if there is a point $x \in \overline{\Sigma} \cap \varphi(M)$, then there is a sequence of disks $D_k \subset \Sigma$ converging to a disk $D \subset \overline{\Sigma}$ containing x. These disks D_k intersect $\varphi(M)$ in a sequence of points of $\varphi(\Gamma)$ converging to x. Since $\varphi(\Gamma)$ is closed, ($\varphi_{|\Gamma}$ is proper), x is in $\varphi(\Gamma)$. $\varphi(\Gamma) \subset \varphi(M) \cap \overline{\Sigma}$ is obvious. If $\mathrm{Lim}\ \varphi \neq \emptyset$, by Theorem (1.1) $\mathrm{Lim}\ \varphi$ intersects Σ. By the same reasoning as above, $\mathrm{Lim}\ \varphi \cap \Sigma \subset \varphi(\Gamma)$ and $\varphi_{|\Gamma}$ is not proper, a contradiction.

Proof of Corollary (1.4). Set $\Sigma = M_1$. If $\Gamma = M_1 \cap M_2 = \emptyset$ then Γ is proper and M_2 is also proper by Case i) of Corollary (1.3). By the Strong Half-Space Theorem they are parallel planes.

4. Proof of Theorem 1.5

Let $\varphi : M \hookrightarrow N$ be a complete minimal hypersurface with scalar curvature bounded from below in a complete Riemannian manifold N of bounded geometry. Recall (Lemma 2.3) that for each $p \in \mathrm{Lim}\ \varphi$ there exists a complete minimal hypersurface $S \subset \mathrm{Lim}\ \varphi$ with scalar curvature bounded from below passing through p, and a sequence $p_k \in \varphi(M)$ converging (up to a subsequence) to p, moreover for each compact set $C_p \subset S$ containing p there is a sequence of compacts $C_k \subset \varphi(M)$ containing p_k converging uniformly to C_p. Such a hypersurface S is called a leaf passing through p. In this section we shall prove Theorem 1.5.

Theorem 1.5. *Let* $\varphi : M^n \hookrightarrow N^{n+1}$ *be a complete minimal immersed hypersurface with scalar curvature bounded from below in a complete dimensional Riemannian manifold* N *of bounded geometry. Suppose in addition that* N *has nonnegative Ricci curvature* $\mathrm{Ric}_N \geq 0$*. Then* φ *is proper or every orientable leaf* $S \subset \mathrm{Lim}\ \varphi$ *such that* $S \cap \varphi(M) = \emptyset$ *is stable. Moreover, if* S *is compact then* S *is totally geodesic and the Ricci curvature of* N *is identically zero in the normal directions to* S*.*

Remark 4.1. For $n = 2$, Schoen shows in [14] that S is totally geodesic if it is noncompact. The proof of this result is close to parts of proofs done by Fisher-Colbrie-Schoen [6]. We will include it here for the sake of completeness.

Proof. Suppose that $\mathrm{Lim}\ \varphi \neq \emptyset$, that is φ is not proper. Let $S \subset \mathrm{Lim}\ \varphi$ be an orientable leaf such that $S \cap \varphi(M) = \emptyset$ hence S has no self intersections. Let $C \subset S$ be a compact and proper subset of S and $T_\epsilon(C)$ an embedded ϵ-tubular neighborhood of C in N. There exists a sequence of compact sets $C_k \subset \varphi(M)$ converging uniformly to C. We may assume that for $k \geq k_0$ the sets C_k are injectively immersed and $C_k \subset T_\epsilon(C)$. Passing to a subsequence if necessary we may assume that $\{C_k\}$ converges to C by one side of C. Let U_ϵ be this side and ν be a continuous unit normal (to S) vector field on S pointing towards U_ϵ. Now (following Ros [11]) we let L be the Jacobi operator on S, i.e., $L = \Delta + \mathrm{Ric}(\nu) + |A|^2$, where Δ is the Laplacian of S, $\mathrm{Ric}(\nu)$ is the Ricci curvature of N in the direction ν and $|A|$ is the norm of the second fundamental form of $S \subset N$. Take a larger compact set C' containing C properly and consider a converging sequence of compact sets also denoted by C_k converging to C'. We may assume that the only solution of $L\,v = 0$ on C' and $v = 0$ on $\partial\,C'$ is the function $v \equiv 0$. Therefore, there exists a function $u \in C^\infty(C')$ such that $L\,u = 1$ on C' and $u = 0$ on $\partial\,C'$. The mean curvature $H(t)$, $|t| < \epsilon$, of the immersions $\psi_t : C' \to \mathbb{R}^3$, $\psi_t(x) = \exp_x(t\,u(x)\,\nu(x))$ for all $x \in C'$, has derivative at $t = 0$ given by $2H'(0) = L\,u = 1$ on C'. Thus if ϵ is small, $H(t) > 0$ on C', $0 < t < \epsilon$. If u is positive at some interior point of C then $\psi_t(C')$, $t < \epsilon$, has a tangency point with some C_k and this is not allowed by the maximum principle, since C_k is a minimal surface and the mean curvature is positive with respect to the vector pointing to that direction. So $u \leq 0$. If $u(q) = 0$, $q \in \mathrm{int}(C')$, by the same reasons as above we have that $u \equiv 0$, and this is impossible. Thus $u < 0$ in the interior of C' and $u = 0$ on the $\partial C'$. Setting $w = -u$ we have that w is a positive function on the interior of C' and $L\,w \leq 0$. In particular w restricted to the boundary of C is a positive function. Let u_1 be the first eigenfunction of C, i.e., $L\,u_1 = -\lambda_1(C)u_1$. Suppose by contradiction that $\lambda_1(C) < 0$. Let $h = w - t\,u_1 > 0$ for some small $t > 0$. We have that $L\,h = L\,w - tL\,u \leq 0 + t\lambda_1\,u_1 < 0$ on C. Then $\Delta\,h < 0$ on C and h has a minimum in the interior. By the maximum principle h is constant. Choosing t in a way that this minimum is zero we have a contradiction. This shows that $\lambda_1(C)$ is positive and thus C is stable.

If S is not compact then there exists an exhaustion of S by compact sets, each one stable. Thus S is stable. When S is compact we will need the following theorem due to Fisher-Colbrie-Schoen [6].

Theorem 4.2 (Fisher-Colbrie-Schoen). *Let (M, ds^2) be a closed Riemannian manifold and let q be a smooth function on M. Given any proper domain D in M, let $\lambda_1(D) < \lambda_2(D) \leq \lambda_3(D) \leq \ldots$ be the sequence of eigenvalues of $\Delta - q$ acting on functions that vanish on ∂D. If $\lambda_1(D) > 0$ for all proper domains D, then there exists a positive function g satisfying the equation $\Delta g - q\,g = 0$ in M.*

This theorem is a part of Theorem 1 of [6] that is valid for closed Riemannian manifolds. Every proper compact domain $C \subset S$ is stable and then the first eigenvalue $\lambda_1(C)$ is positive for the stability operator $\Delta + \operatorname{Ric}\nu + |A|^2 = \Delta - q$. Here A is the second fundamental form of $S \subset N$ and ν is a unit vector field in S and normal to S; thus by Theorem 4.2, there exists a positive function g in S satisfying $\Delta g - q\,g = 0$. Therefore,

$$\int_S \Delta g - \int_S q\,g = 0 \Rightarrow \int_S q\,g = 0 \Rightarrow q = 0 \Rightarrow |A| = 0 \text{ and } \operatorname{Ric}\nu = 0.$$

Then S is totally geodesic and the Ricci curvature is zero in the normal directions to S.

The stability operator is then $L = \Delta$, acting on functions $f : S \to \mathbb{R}$ with $\int_S f = 0$. Now suppose that S is not stable. Then $\lambda_1(S) < 0$ and $\Delta f + \lambda_1(S) f = 0$ in S for some function $f : S \to \mathbb{R}$ with $\int_S f = 0$. Let $D_f = \{x \in S : f(x) > 0\}$ be the nodal set of f. Thus

$$\begin{aligned}\lambda_1(D_f) = {} & \inf\{\textstyle\int_{D_f} u\Delta u / \int u^2; \operatorname{supp} u \subset D_f\} \leq \int_{D_f} f\Delta f / \int f^2 \\ = {} & -\textstyle\int_{D_f} |\nabla f|^2 / \int f^2 < 0.\end{aligned}$$

This contradicts the fact that $\lambda_1(C) > 0$ (stability of compact proper subsets) of any compact subset. Therefore S is stable. For $n = 2$, Schoen [14] has shown that a complete (non compact) stable minimal surface in a 3-dimensional Riemannian manifold with nonnegative Ricci curvature is totally geodesic.

Now we can prove Corollary 1.6 as follows. Suppose that $\varphi : M \hookrightarrow N$ is a complete noncompact minimally and injectively immersed surface M with sectional curvature bounded from below into a 3-dimensional compact Riemannian manifold N with positive Ricci curvature. Since M is not compact, $\mathrm{Lim}\ \varphi \neq \emptyset$. Let $S \subset Lim(S)$. By hypotheses, $\varphi(M)$ has no self intersections, hence S has no self intersections either and $\varphi(M) \cap S = \emptyset$. Since $\mathrm{Ric}_N > 0$ then the first Betti number of N, $b_1(N) = 0$. Thus S is orientable. By Theorem 1.5 S is stable and totally geodesic. By Corollary 3 of [14] S is compact; in fact S is conformally equivalent to the sphere $\mathbb{S}^2$. Again, by Theorem 1.5, the Ricci curvature $\mathrm{Ric}_N(\nu) \equiv 0$ in the normal directions to S, a contradiction.

Suppose that N is not compact and M is not proper, then $\mathrm{Lim}\ \varphi \neq \emptyset$. There is a leaf $S \subset \mathrm{Lim}\ \varphi$ such that $S \cap \varphi(M) = \emptyset$ since M is injectively immersed. By Corollary 3 of [14], S is compact and thus the Ricci curvature in the normal directions are zero, contradiction. Therefore, $\mathrm{Lim}\ \varphi = \emptyset$ and M is proper.

5. An application of the maximum principle

In this section we present a particular case of Theorem 1.1 to show a way that the maximum principle can be applied to nonproper minimal immersions.

Corollary 5.1. *Let $\varphi : M \hookrightarrow \mathbb{R}^3$ be a complete minimal immersed surface in $\mathbb{R}^3$ with bounded sectional curvature and let C be any catenoid in $\mathbb{R}^3$. Then $M \cap C \neq \emptyset$.*

Suppose by contradiction that $M \cap C = \emptyset$. We will assume that $\mathrm{Lim}\ \varphi \neq \emptyset$ (otherwise the Strong Half-Space Theorem implies the claim) and it is connected. Observe that $\mathrm{Lim}\ \varphi \neq C$ because C is not stable (see Theorem 1.5). In fact, we may suppose that $\mathrm{Lim}\ \varphi \cap C = \emptyset$, because otherwise it would imply that M intersects C. So we have that M neither intersect C nor accumulates on C. Let $\Omega(C)$ be the simply connected open region of $\mathbb{R}^3$ whose boundary is C. We may assume that C does not intersects the x_3-axis (after a rotation of $\mathbb{R}^3$). Let B_R be a closed ball in $\mathbb{R}^3$ centered at the origin and radius R such that $B_R \cap \mathrm{Lim}\ \varphi \neq \emptyset$. Suppose first that $\mathrm{Lim}\ \varphi \subset \Omega(C)$. If $\mathrm{Lim}\ \varphi$ does not intersect the plane $x_3 = 0$ there is a point $q \in B_R \cap \mathrm{Lim}\ \varphi$ closest to the plane $\{x_3 = 0\}$ with positive distance. Move the plane $\{x_3 = 0\}$ parallely, (i.e the planes are $\{x_3 = t\}$) towards q till it touches the first point p (possibly q) in the compact set $B_R \cap \mathrm{Lim}\ \varphi$ say, at $x_3 = t_0$. By Theorem 1.5 there is a complete minimal surface $S \subset \mathrm{Lim}\ \varphi$ passing through p. This minimal surface S touches the plane $\{x_3 = t_0\}$ at p but does not cross it because p is the closest point in $B_R \cap \mathrm{Lim}\ \varphi$ to the plane $\{x_3 = 0\}$ and a piece of S is still in the compact $B_R \cap \mathrm{Lim}\ \varphi$. By the maximum principle, S is the plane $\{x_3 = t_0\}$ and it must intersect C. So, if $\mathrm{Lim}\ \varphi \subset \Omega(C)$, then it does intersect the plane $\{x_3 = 0\}$, in fact the reasoning above shows that it intersects all the planes $\{x_3 = t\}$. Recalling that $B_R \cap \mathrm{Lim}\ \varphi$ is compact and does not touch $B_R \cap C$, we then make a homothety of C, shrinking the catenoid C to another catenoid $\widetilde{C}$ till it touches a first point $\widetilde{p} \in B_R \cap \mathrm{Lim}\ \varphi$. With the same reasoning the maximum principle applies and we have that $\mathrm{Lim}\ \varphi \cap C \neq \emptyset$. Therefore, $\mathrm{Lim}\ \varphi \subset \left[\mathbb{R}^3 \setminus \overline{\Omega}(C)\right]$. In this case there is a point $\hat{q} \in B_R \cap \mathrm{Lim}\ \varphi$ closest to the x_3-axis. We make a homothety of C to enlarge it to another catenoid $\hat{C}$ that first touches at a point $\hat{p} \in B_R \cap \mathrm{Lim}\ \varphi$. Again, the minimal surface $S(\hat{p})$ would coincide to $\hat{C}$ and would intersect C. In any of the cases we have a contradiction to the hypotheses $\mathrm{Lim}\ \varphi \neq \emptyset$ and $\mathrm{Lim}\ \varphi \cap C = \emptyset$.

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